L^q HARMONIC FUNCTIONS ON GRAPHS

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Abstract. We prove an analogue of Yau's Caccioppoli-type inequality for nonnegative subharmonic functions on graphs. We then obtain a Liouville theorem for harmonic or non-negative subharmonic functions of class L^q, $1 \leq q < \infty$, on any graph, and a quantitative version for $q > 1$. Also, we provide counterexamples for Liouville theorems for $0 < q < 1$.

1. Introduction

In 1976, Yau [Yau76] proved an L^q $(1 < q < \infty)$ Liouville theorem for harmonic functions on complete Riemannian manifolds. Yau's theorem says that there doesn't exist any nonconstant L^q $(1 < q < \infty)$ harmonic functions on any complete Riemannian manifold M. This result is quite remarkable since it does not require any assumption besides the – obviously necessary – completeness on the underlying manifold.

Karp [Kar82] then found a quantitative version of Yau's L^q Liouville theorem. Let f be a nonconstant nonnegative subharmonic function on M. Then

$$\liminf_{R\to\infty} \frac{1}{R^2} \int_{B_R(p)} f^q d\mathrm{vol} = \infty, \tag{1}$$

where $B_R(p)$ is the geodesic ball centered at p of radius R and $1 < q < \infty$. After that, Li-Schoen [LS84] proved an L^q mean value inequality for subharmonic functions on manifolds with proper curvature conditions, which implies the L^q Liouville theorem for such manifolds.

In 1997, Rigoli-Salvatori-Vignati [RSV97] generalized Karp's version of the L^q Liouville theorem to the graph setting. Under the assumption of uniformly bounded degree for the graph, they proved the analogue of (1) (with sums in place of integrals) for the case $q \geq 2$. The case $1 < q < 2$ was left open. In this paper, we use an idea of Yau, the L^q Caccioppoli-type inequality (see Theorem 3.1), to prove a theorem that resolves that case.

In order to introduce our setting for graphs, let $G = (V, E)$ be an infinite, connected, locally finite, weighted graph. Each edge $e \in E$ carries a positive weight μ_e, and for each vertex $x \in V$ this then induces the positive weight $\mu_x = \sum_{y\sim x} \mu_{xy}$, called the vertex degree of x, where $y \sim x$ means that they are neighbors. We denote by $B_R(p)$ the closed ball centered at p of radius R in G where the distance between two vertices is given by the minimal number of edges to be traversed when going from one to the other.

The research leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013) / ERC grant agreement n° 267087.

The (normalized) Laplace operator is defined as

$$\Delta f(x) = \frac{1}{\mu_x} \sum_{y \sim x} \mu_{xy} \left(f(y) - f(x) \right).$$

A function f is called harmonic (subharmonic) if $\Delta f = 0 (\geq 0)$.

We can now formulate our main result.

Theorem 1.1. *Let f be a nonnegative subharmonic function on the weighted graph G. Then, either f is constant or, for any $q \in (1, \infty)$,*

$$\liminf_{R \to \infty} \frac{1}{R^2} \sum_{B_R(p)} f^q(x) \mu_x = \infty. \tag{2}$$

Note that in our theorem, in contrast to [RSV97], we do not need to assume any uniform upper and lower bounds of the vertex degree of the graph. This is consistent with the intuition that nonconstant harmonic functions on infinite graphs of finite volume grow extremely fast. We will give an example to show that (2) is false for the case $q = 1$, see Remark 4.1.

It is well-known that for graphs with $\mu_x \geq \mu_0 > 0$ for all $x \in V$, the L^q $(0 < q < \infty)$ Liouville theorem is an easy consequence of the maximum principle for subharmonic functions, see Theorem 2.1. Thus, the only interesting case of the L^q Liouville theorem concerns general weighted graphs. In the case of Riemannian manifolds, soon after Yau's L^q $(1 < q < \infty)$ Liouville theorem, Chung [Chu83] provided an example to show that there is no L^1 Liouville theorem. Counterexamples for an L^q $(0 < q \leq 1)$ Liouville theorem were then given in Li-Schoen [LS84]. That is why Li-Schoen [LS84] proved the L^q $(0 < q < \infty)$ Liouville theorem under an additional curvature assumptions. Surprisingly, we can adopt an idea of Li [Li84, Li12] to prove the L^q $(1 \leq q < \infty)$ Liouville theorem, even for the borderline case $q = 1$. Although the curvature assumptions are necessary in the Riemannian case by Li [Li84], on graphs we don't need any such curvature-like assumptions. There are other generalizations of the L^q Liouville theorem, for instance, by Sturm [Stu94] to strongly local regular Dirichlet forms (not including graphs, $q \neq 1$) and Masamune [Mas09] to graphs with different weights $(2 \leq q \in \mathbb{N})$. We can show

Theorem 1.2. *For any graph G, there doesn't exist any nonconstant L^q harmonic (nonnegative subharmonic) function for $q \in [1, \infty)$.*

We give examples of a large class of graphs with infinite volume that don't satisfy the L^q Liouville theorem for any $0 < q < 1$, see Example 4.2.

As an application, we study the L^q Liouville theorem for higher order elliptic operators on graphs where the maximum principle is no longer available.

The organization of the paper is as follows: The basic facts on graphs are collected in Sect. 2, the next section contains the proof of Yau's Caccioppoli-type inequality and the solution to the problem of Rigoli-Salvatori-Vignati [RSV97], and the last section is devoted to the L^1 Liouville theorem on graphs and an application to higher order operators.

2. Preliminaries

Let $G = (V, E)$ be an infinite, connected, locally finite, weighted graph (see e.g. [Chu97, Gri09] for definitions). $G = (V, E)$ is a weighted graph with positive and symmetric edge weights $\mu_{xy} > 0$ for any $xy \in E$. For convenience, we extend the

edge weight to $V \times V$ by $\mu_{xy} = 0$ for $xy \notin E$. The graph G may have self-loops, i.e. $xx \in E$ (or $\mu_{xx} > 0$) but w.l.o.g. we exclude multiple edges because they can be implicitly encoded in the edge weights μ_{xy}. Define the measure on V as $\mu_x = \sum_y \mu_{xy}$ for $x \in V$. Let us denote by $\mu(G) := \sum_x \mu_x$ the total volume of the graph G. There are many interesting graphs of finite volume. The Laplace operator on G is defined as

$$\Delta f(x) = \sum_y \frac{\mu_{xy}}{\mu_x} \left(f(y) - f(x)\right).$$

The transition operator associated to the random walk on the graph is defined as $Pf(x) = \sum_y P(x,y)f(y)$ where $P(x,y) = \frac{\mu_{xy}}{\mu_x}$. Obiviously, $\Delta = P - I$ where I is the identity operator.

There is a natural (combinatorial) distance function d on the graph, simply counting the minimal number of edges separating two vertices. We denote by $B_R(p) := \{x \in V : d(x,p) \leq R\}$ the closed ball centered at p of radius R. For any subset $\Omega \subset G$, we denote by $d(x,\Omega) := \min\{d(x,y) : y \in \Omega\}$ the distance to Ω, by $\partial\Omega := \{y \in G : d(x,\Omega) = 1\}$ the boundary of Ω. A function $f : \Omega \cup \partial\Omega \to \mathbb{R}$ is called *harmonic (subharmonic, superharmonic)* on Ω if $\Delta f(x) = 0$ $(\geq 0, \leq 0)$ for all $x \in \Omega$. For any function f on G we denote the L^q norm of f by $\|f\|_q := (\sum_x |f(x)|^q \mu_x)^{1/q}$, $q \in (0,\infty)$.

For our difference operators, we need orientations. We choose an orientation for the edge set E, that is, $e = xy$ means that the edge e starts at x and ends at y. Let us denote by $\mathbb{R}^V := \{f : V \to \mathbb{R}\}$ (resp. $\mathbb{R}^E$) the set of all functions on V (resp. on E), by $C_c(V)$ (resp. $C_c(E)$) the space of compact supported functions defined on V (resp. on E). We define inner products on $C_c(V)$ and $C_c(E)$ as

$$\begin{aligned} \langle f, g\rangle &= \sum_{x\in V} f(x)g(x)\mu_x, \\ \langle u, v\rangle &= \sum_{e\in E} u(e)v(e)\mu_e, \end{aligned}$$

where $f, g \in C_c(V)$ and $u, v \in C_c(E)$. For any $f \in \mathbb{R}^V$, the pointwise gradient $\nabla f \in \mathbb{R}^E$ is defined as $\nabla f(e) = \nabla_e f := f(y) - f(x)$ for all $e = xy \in E$. A very useful formula reads as, for any $e = xy$,

$$\nabla_e(fg) = f(x)\nabla_e g + \nabla_e f g(y).$$

In addition, we have Green's formula (see e.g. [Gri09]), for $f \in \mathbb{R}^V$ and $g \in C_c(V)$,

$$\langle \Delta f, g\rangle = -\langle \nabla f, \nabla g\rangle.$$

In the following, we mean by $e \subset A$ for some subset $A \subset V$ that both vertices of the edge e are contained in A.

The graph is called *non-degenerate* if $\mu_x \geq \mu_0 > 0$ for all $x \in V$. As a well-known result, we will see that there are no nontrivial L^q nonnegative subharmonic functions on non-degenerate graphs for any $q \in (0,\infty)$. This follows from the maximum principle for subharmonic functions and the uniform lower bound of the measure of a non-degenerate graph.

Lemma 2.1 (Maximum principle)**.** *Let Ω be a finite connected subset of G and f be subharmonic on Ω. Then*

$$\max_\Omega f \leq \max_{\partial\Omega} f, \tag{3}$$

where the equality holds iff f is constant on $\Omega \cup \partial\Omega$.

Proof. This follows from the definition of subharmonic functions and the connectedness of Ω (see [Gri09]). □

Theorem 2.1. *Let G be a non-degenerate graph. Then there doesn't exist any nontrivial L^q nonnegative subharmonic functions on G, $q \in (0, \infty)$.*

Proof. Let f be an L^q nonnegative subharmonic function. Since G is a non-degenerate graph and $f \in L^q$, we have $f(x) \to 0$ as $x \to \infty$. By the maximum principle, this yields that $|f| \le \epsilon$ for any $\epsilon > 0$. That is, $f \equiv 0$. □

If G is not non-degenerate, we cannot apply the maximum principle as above. In fact, there are L^q harmonic functions on some graphs which are unbounded at infinity.

3. L^q subharmonic and harmonic functions

In this section, we generalize Yau's L^q Caccioppoli-type inequality on manifolds (see Lemma 7.1 in [Li12]) to graphs. This will imply the L^q Liouville theorem for harmonic functions when $q \in (1, \infty)$. Moreover, using this estimate, we shall prove the main Theorem 1.1.

Theorem 3.1 (Caccioppoli-type inequality)**.** *Let f be a nonnegative subharmonic function on the weighted graph G. Then for any $1 < q < \infty$, $0 < r < R-1$, $r, R \in \mathbb{N}$*

$$\sum_{e=xy\subset B_r} \mu_{xy} |\nabla_{xy} f|^2 \min\{f^{q-2}(x), f^{q-2}(y)\} \le \frac{C}{(R-r)^2} \sum_{B_R \setminus B_r} f^q(x)\mu_x. \tag{4}$$

Remark 3.1. The classical Caccioppoli inequality is the case $q = 2$. On graphs, see e.g. [CG98, HS97, LX10].

Remark 3.2. We take the convention that $0 \cdot \infty = 0$ on the LHS of (4) for the case $1 < q < 2$ because it suffices to consider positive subharmonic functions by setting $f_\epsilon = f + \epsilon$ $(\epsilon \to 0)$.

Proof. Fix a point $p \in G$ and denote the distance function to p by $r(x) = d(x, p)$. We denote by $B_r := B_r(p)$ the closed ball centered at p of radius r. Let us choose a test function φ satisfying

$$\varphi(x) = \begin{cases} 1, & r(x) \le r+1, \\ \frac{R-r(x)}{R-r-1}, & r+1 \le r(x) \le R, \\ 0, & R \le r(x). \end{cases}$$

Then $\mathrm{supp}\varphi \subset B_R$, $\nabla_e \varphi \neq 0$ only if $e \subset B_R \setminus B_r$, and $|\nabla_e \varphi| \le \frac{2}{R-r}$ for all $e \in E$.

For the fixed subharmonic function f, we choose a particular orientation of E such that for any $e \in E$, $e = xy$ where $f(y) \ge f(x)$. We divide the proof into two cases.

Case 1. $2 \le q < \infty$. Using $\varphi^2 f^{q-1}$ as the test function, we have

$$\begin{aligned}
0 &\le \langle \Delta f, \varphi^2 f^{q-1}\rangle = -\langle \nabla f, \nabla(\varphi^2 f^{q-1})\rangle \\
&= -\sum_{e\in E} \mu_e \nabla_e f \nabla_e(\varphi^2 f^{q-1}) \\
&= -\sum_{e=xy\in E} \mu_e \nabla_e f \left[\varphi^2(x)\nabla_e(f^{q-1}) + \nabla_e\varphi(\varphi(x)+\varphi(y))f^{q-1}(y)\right] \\
&\quad (\text{using } f^{q-1}(y) = \nabla_{xy}(f^{q-1}) + f^{q-1}(x)) \\
&= -\sum_e \mu_e \nabla_e f \nabla_e(f^{q-1})\varphi^2(y) - \sum_e \mu_e \nabla_e f \nabla_e\varphi(\varphi(x)+\varphi(y))f^{q-1}(x) \\
&\quad (\text{by the mean value inequality } \nabla_e(f^{q-1}) \ge (q-1)\nabla_e f f^{q-2}(x)) \\
&\le -(q-1)\sum_e \mu_e|\nabla_e f|^2 f^{q-2}(x)\varphi^2(y) - \sum_e \mu_e \nabla_e f \nabla_e\varphi(\varphi(x)+\varphi(y))f^{q-1}(x) \qquad (5)\\
&= -(q-1)\sum_e \mu_e|\nabla_e f|^2 f^{q-2}(x)\varphi^2(y) - 2\sum_e \mu_e \nabla_e f \nabla_e\varphi\varphi(y)f^{q-1}(x) \\
&\quad + \sum_e \mu_e \nabla_e f|\nabla_e\varphi|^2 f^{q-1}(x).
\end{aligned}$$

Using Young's inequality for the second term in the last inequality,

$$2\nabla_e f|\nabla_e\varphi|\varphi(y)f^{q-1}(x) \le \frac{q-1}{2}|\nabla_e f|^2 f^{q-2}(x)\varphi^2(y) + C|\nabla_e\varphi|^2 f^q(x),$$

we obtain that

$$\begin{aligned}
0 &\le -(q-1)/2\sum_e \mu_e|\nabla_e f|^2 f^{q-2}(x)\varphi^2(y) + C\sum_e \mu_e|\nabla_e\varphi|^2 f^q(x) \\
&\quad + \sum_e \mu_e \nabla_e f|\nabla_e\varphi|^2 f^{q-1}(x) \\
&= I + II + III. \qquad (6)
\end{aligned}$$

By the choice of φ we have

$$II \le \frac{C}{(R-r)^2}\sum_{e\subset B_R\setminus B_r} \mu_e f^q(x) \le \frac{C}{(R-r)^2}\sum_{x\in B_R\setminus B_r} \mu_x f^q(x),$$

and

$$\begin{aligned}
III &\le \frac{C}{(R-r)^2}\sum_{e\subset B_R\setminus B_r} \mu_e(f(y)-f(x))f^{q-1}(x) \le \frac{C}{(R-r)^2}\sum_{e\subset B_R\setminus B_r} \mu_e f(y)f^{q-1}(x) \\
&\le \frac{C}{(R-r)^2}\sum_{e\subset B_R\setminus B_r} \mu_e f^q(y) \le \frac{C}{(R-r)^2}\sum_{x\in B_R\setminus B_r} \mu_x f^q(x).
\end{aligned}$$

Hence by (6),

$$\begin{aligned}
\sum_{e=xy\subset B_r} \mu_e|\nabla_e f|^2 \min\{f^{q-2}(x), f^{q-2}(y)\} &\le \sum_e \mu_e|\nabla_e f|^2 f^{q-2}(x)\varphi^2(y) \\
&\le \frac{C}{(R-r)^2}\sum_{B_R\setminus B_r} \mu_x f^q(x).
\end{aligned}$$

Case 2. $1 < q \leq 2$. The only difference here is that by the mean value inequality $\nabla_{xy}(f^{q-1}) \geq (q-1)\nabla_{xy} f f^{q-2}(y)$. By a similar calculation as in Case 1, we have

$$\begin{aligned}
0 &\leq \langle \Delta f, \varphi^2 f^{q-1}\rangle = -\sum_{e\in E}\mu_e \nabla_e f \nabla_e(\varphi^2 f^{q-1}) \\
&= -\sum_{e=xy\in E}\mu_e\nabla_e f\left[\nabla_e(f^{q-1})\varphi^2(y) + f^{q-1}(x)\nabla_e\varphi(\varphi(x)+\varphi(y))\right] \\
&= -\sum_e \mu_e\nabla_e f\left[\nabla_e(f^{q-1})\varphi^2(x) + f^{q-1}(y)\nabla_e\varphi(\varphi(x)+\varphi(y))\right] \\
&\leq -(q-1)\sum_e \mu_e|\nabla_e f|^2 f^{q-2}(y)\varphi^2(x) - 2\sum_e \mu_e\nabla_e f\nabla_e\varphi\varphi(x)f^{q-1}(y) \\
&\quad -\sum_e \mu_e\nabla_e f|\nabla_e\varphi|^2 f^{q-1}(y) \\
&\leq -(q-1)\sum_e \mu_e|\nabla_e f|^2 f^{q-2}(y)\varphi^2(x) - 2\sum_e \mu_e\nabla_e f\nabla_e\varphi\varphi(x)f^{q-1}(y). \quad (7)
\end{aligned}$$

Using Young's inequality for the second term and the same argument as before, we have

$$\sum_e \mu_e|\nabla_e f|^2 f^{q-2}(y)\varphi^2(x) \leq \frac{C}{(R-r)^2}\sum_{B_R\setminus B_r}\mu_x f^q(x).$$

This proves

$$\sum_{e=xy\subset B_r}\mu_e|\nabla_e f|^2\min\{f^{q-2}(x), f^{q-2}(y)\} \leq \frac{C}{(R-r)^2}\sum_{B_R\setminus B_r}\mu_x f^q(x)$$

which is (4).

□

By this Caccioppoli-type inequality, we can prove the L^q Liouville theorem for nonnegative subharmonic functions when $q \in (1,\infty)$.

Corollary 3.1. *For any graph G, there doesn't exist any nonconstant L^q harmonic (nonnegative subharmonic) function for $q \in (1,\infty)$.*

Proof. For any harmonic function f, $|f|$ is subharmonic. Hence, it suffices to prove the corollary for nonnegative subharmonic functions. Suppose f is an L^q nonnegative subharmonic function on G. We apply Theorem 3.1 by setting $R = 2r$. Since the RHS of (4) tends to zero as $r \to \infty$, we have for any $e = xy \in E$

$$|\nabla_e f|\min\{f^{q-2}(x), f^{q-2}(y)\} = 0. \quad (8)$$

Now we claim that $|\nabla_e f| = 0$ for any $e \in E$. Suppose not, then there exists an $e \in E$ such that $|\nabla_e f| \neq 0$, w.l.o.g., we may assume $e = xy$ and $f(y) > f(x)$. By the equation (8), $\min\{f^{q-2}(x), f^{q-2}(y)\} = 0$. For $1 < q < 2$, $\min\{f^{q-2}(x), f^{q-2}(y)\} > 0$ which yields a contradiction. For $2 \leq q < \infty$, we have $f(x) = 0 < f(y)$. For any $z \sim y$ and $z \neq x$, using the equation (8) for yz, we have $f(z) = 0$ or $f(z) = f(y)$. Noting that $f(x) < f(y)$, the subharmonicity of f at y implies that

$$f(y) \leq \sum_w \frac{\mu_{yw}}{\mu_y} f(w) < f(y).$$

This is a contradiction which proves the claim. Then f is constant. □

Now we prove the main Theorem 1.1 which settles a question in [RSV97].

Proof of Theorem 1.1. We divide the proof into two cases, $1 < q \le 2$ and $2 \le q < \infty$. In the following, we assume $\liminf_{R\to\infty} \frac{1}{R^2}\sum_{B_R} f^q(x)\mu_x < \infty$ and show that f is constant. For any $r, R \in \mathbb{N}$, $r+1 < R$, we define the test function as

$$\varphi(x) = \varphi_{r,R}(x) = \begin{cases} 1, & r(x) \le r+1, \\ \frac{R-r(x)}{R-r-1}, & r+1 \le r(x) \le R, \\ 0, & R \le r(x). \end{cases}$$

Then $\nabla_e \varphi \neq 0$ only if $e \subset B_R \setminus B_r$, and $|\nabla_e\varphi| \le \frac{2}{R-r}$ for all $e \in E$. In addition, $\varphi(x) \le 2\varphi(y)$ for any $e = xy \not\subset B_R \setminus B_{R-2}$.

Case 1. $1 < q \le 2$. Using the test function $\varphi_{r,R}$ in (7), we have

$$\sum_{e\subset B_R} \mu_e |\nabla_e f|^2 f^{q-2}(y)\varphi^2(x) \le C(q) \sum_{e\subset B_R\setminus B_r} \mu_e \nabla_e f |\nabla_e \varphi| \varphi(x) f^{q-1}(y).$$

Hence by the Hölder inequality,

$$\begin{aligned} &\left(\sum_{e\subset B_R} \mu_e |\nabla_e f|^2 f^{q-2}(y)\varphi^2(x)\right)^2 \\ \le\ & C\left(\sum_{e\subset B_R\setminus B_r} \mu_e |\nabla_e f|^2 f^{q-2}(y)\varphi^2(x)\right)\left(\sum_{e\subset B_R\setminus B_r} \mu_e |\nabla_e \varphi|^2 f^q(y)\right) \\ \le\ & C\left(\Big(\sum_{e\subset B_R} - \sum_{e\subset B_r}\Big) \mu_e |\nabla_e f|^2 f^{q-2}(y)\varphi^2(x)\right)\left(\frac{C}{(R-r)^2}\sum_{x\in B_R\setminus B_r} f^q(x)\mu_x\right). \end{aligned} \tag{9}$$

Since we assume $\liminf_{R\to\infty} \frac{1}{R^2}\sum_{B_R} f^q(x)\mu_x < \infty$, there exists a sequence $\{R_i\}_{i=1}^{\infty}$, $R_i \to \infty$, such that $R_{i+1} \ge 2R_i$ and $\frac{1}{R_i^2}\sum_{B_{R_i}} f^q(x)\mu_x \le K < \infty$ for all $i \in \mathbb{N}$. We define

$$\begin{aligned} A_i &:= \frac{1}{R_i^2}\sum_{B_{R_i}} f^q(x)\mu_x, \\ \varphi_i &:= \varphi_{R_i,R_{i+1}}(x), \\ Q_{i+1} &:= \sum_{e\subset B_{R_{i+1}}} \mu_e |\nabla_e f|^2 f^{q-2}(y)\varphi_i^2(x). \end{aligned} \tag{10}$$

Now by setting $R = R_{i+1}$ and $r = R_i$, the inequality (9) reads as

$$\begin{aligned} Q_{i+1}^2 &\le C(Q_{i+1} - Q_i)\frac{R_{i+1}^2}{(R_{i+1}-R_i)^2} A_{i+1} \\ &\le C(Q_{i+1} - Q_i) A_{i+1} \quad (\text{ by } R_{i+1} \ge 2R_i). \end{aligned} \tag{11}$$

Since $A_i \le K$ for any $i \in \mathbb{N}$,

$$Q_{i+1}^2 \le C Q_{i+1} K.$$

This implies that $Q_{i+1} \le CK$ for all i. On the other hand, (11) implies that

$$Q_{i+1}^2 \le CK(Q_{i+1} - Q_i).$$

Summing over i in the above inequality we have, for any integer N,

$$\sum_{i=1}^{N} Q_{i+1}^2 \leq CK(Q_{N+1} - Q_1) \leq (CK)^2.$$

Hence $Q_i \to 0$ as $i \to \infty$.

Case 2. $2 \leq q < \infty$. This argument follows from the idea of [RSV97]. Using the test function $\varphi_{r,R}$ in (5), we have

$$\begin{aligned}
&\sum_{e \subset B_R} \mu_e |\nabla_e f|^2 f^{q-2}(x)\varphi^2(y) \leq C(q) \sum_{e \subset B_R \setminus B_r} \mu_e \nabla_e f |\nabla_e \varphi| (\varphi(x) + \varphi(y)) f^{q-1}(x) \\
&\quad (\text{ by } \varphi(x) \leq 2\varphi(y) \text{ for } e = xy \not\subset B_R \setminus B_{R-2}) \\
\leq\ & 2C \sum_{e \subset B_R \setminus B_r} \mu_e \nabla_e f |\nabla_e \varphi| \varphi(y) f^{q-1}(x) + C \sum_{e \subset B_R \setminus B_{R-2}} \mu_e \nabla_e f |\nabla_e \varphi|^2 f^{q-1}(x).
\end{aligned}$$

By the Hölder inequality, we have

$$\begin{aligned}
&\left(\sum_{e \subset B_R} \mu_e |\nabla_e f|^2 f^{q-2}(x) \varphi^2(y) \right)^2 \\
\leq\ & C \left(\sum_{e \subset B_R \setminus B_r} \mu_e |\nabla_e f|^2 f^{q-2}(x) \varphi^2(y) \right) \left(\sum_{e \subset B_R \setminus B_r} \mu_e f^q(x) |\nabla_e \varphi|^2 \right) \\
&+ C \left(\sum_{e \subset B_R \setminus B_{R-2}} \mu_e |\nabla_e f|^2 f^{q-2}(x) |\nabla_e \varphi|^2 \right) \left(\sum_{e \subset B_R \setminus B_{R-2}} \mu_e f^q(x) |\nabla_e \varphi|^2 \right) \\
\leq\ & C \left(\sum_{e \subset B_R \setminus B_r} \mu_e |\nabla_e f|^2 f^{q-2}(x) \varphi^2(y) + \frac{C}{(R-r)^2} \sum_{e \subset B_R \setminus B_{R-2}} \mu_e |\nabla_e f|^2 f^{q-2}(x) \right) \\
&\times \left(\frac{C}{(R-r)^2} \sum_{x \in B_R \setminus B_r} \mu_x f^q(x) \right) \qquad (12)
\end{aligned}$$

We keep A_i and φ_i as in (10) and define two more quantities,

$$\begin{aligned}
Q_{i+1} &:= \sum_{e \subset B_{R_{i+1}}} \mu_e |\nabla_e f|^2 f^{q-2}(x) \varphi_i^2(y), \\
\beta_i &:= \frac{C}{(R_{i+1} - R_i)^2} \sum_{e \subset B_{R_{i+1}} \setminus B_{R_{i+1}-2}} \mu_e |\nabla_e f|^2 f^{q-2}(x).
\end{aligned}$$

With these notations, the inequality (12) reads as, by $R_{i+1} \geq 2R_i$,

$$Q_{i+1}^2 \leq CA_{i+1}(Q_{i+1} - Q_i + \beta_i) \leq CK(Q_{i+1} - Q_i + \beta_i). \qquad (13)$$

On the other hand, noting that $\nabla_e f \leq f(y)$,

$$\begin{aligned}
\beta_i &\leq \frac{C}{(R_{i+1}-R_i)^2} \sum_{e\subset B_{R_{i+1}}\setminus B_{R_{i+1}-2}} \mu_e f^q(y) \\
&\leq \frac{C}{(R_{i+1}-R_i)^2} \sum_{x\in B_{R_{i+1}}\setminus B_{R_{i+1}-2}} f^q(x)\mu_x \quad (\text{ by } R_{i+1}\geq 2R_i) \\
&\leq CA_{i+1} \leq CK.
\end{aligned}$$

Hence (13) implies that

$$Q_{i+1}^2 \leq CK(Q_{i+1}+CK).$$

This implies that that Q_i is bounded, i.e. $Q_{i+1} \leq C(K)$. Hence $Q_i \leq Q_{i+1} \uparrow Q < \infty$. By the definition of β_i and Q_{i+1}, we have

$$\beta_i \leq \frac{C}{(R_{i+1}-R_i)^2} Q \leq \frac{CQ}{R_i^2} \to 0, \quad (i\to\infty).$$

Taking the limit $i \to \infty$ in (13), we obtain $Q = 0$. This is the estimate we need for the Case 2.

In both cases, $Q_i \to 0$ as $i \to \infty$. Since $\varphi_i(x) \equiv 1$ for $x \in B_{R_i}$, for any $R > 0$ and sufficiently large $R_i \gg R$ we have

$$\sum_{e\subset B_R} \mu_e |\nabla_e f|^2 \min\{f^{q-2}(x), f^{q-2}(y)\} \leq Q_{i+1} \to 0, \quad (i\to\infty).$$

Hence for each $e = xy \in E$, we have $|\nabla_e f| \min\{f^{q-2}(x), f^{q-2}(y)\} = 0$. A similar argument as in Corollary 3.1 shows that f is constant. □

Remark 3.3. For the case $1 < q \leq 2$, we are now in the situation of (9). We may thus obtain the precise analogues of Theorem 2.2 in Karp [Kar82] and Theorem 1 (b) in Sturm [Stu94]. This is unknown for the case $q > 2$.

4. Borderline case and counterexamples

In this section, we deal with the borderline case, i.e. $q = 1$. We adopt an idea of Li [Li84, Li12] to prove the L^1 Liouville theorem for nonnegative subharmonic functions. In our setting, we don't need any curvature-like assumptions. We shall now complete the proof of Theorem 1.2 by settling the L^1-case.

Proof of Theorem 1.2. For the case $1 < q < \infty$, see Corollary 3.1. We only need to prove the theorem for $q = 1$. Let f be a nonnegative L^1 subharmonic function. We claim that f is harmonic. The subharmonicity of f implies that $\Delta f = (P-I)f \geq 0$, i.e. for all $x \in G$,

$$Pf(x) \geq f(x), \tag{14}$$

where P is the transition operator. Since $f \in L^1(G)$ and $f \geq 0$,

$$\begin{aligned}
\|Pf\|_1 &= \sum_x Pf(x)\mu_x = \sum_{x,y} p(x,y)f(y)\mu_x \\
&= \sum_{x,y} \mu_{xy} f(y) = \sum_{x,y} p(y,x) f(y)\mu_y \\
&= \sum_y \left(\sum_x p(y,x)\right) f(y)\mu_y = \|f\|_1.
\end{aligned}$$

Hence by the monotonicity of (14), we have $Pf(x) = f(x)$ for all $x \in G$. This proves the claim.

For any $a > 0$, we define a function $g := \min\{f, a\}$. Since f is harmonic, a straightforward calculation shows that g is superharmonic, i.e. $\Delta g = (P - I)g \leq 0$. It is easy to see that $0 \leq g \in L^1(G)$ (by $g \leq f$). A similar computation as before yields that $\|Pg\|_1 = \|g\|_1$. The monotonicity of $Pg \leq g$ implies that g is harmonic, i.e. $Pg = g$. Since a is arbitrary, we can prove that f is constant. Suppose that f is not constant, then there exists $x, y \in G$ such that $x \sim y$ and $f(x) \neq f(y)$. Without loss of generality, we may assume $f(x) < f(y)$. Now choose $a = f(x)$. By the harmonicity of f and g,

$$f(x) = g(x) = Pg(x) < Pf(x) = f(x).$$

A contradiction. This proves the theorem. □

Now we give two examples to show that the L^q Liouville theorem is not true for $q \in (0, 1)$. The first example is a graph of finite volume and the second of infinite volume.

Example 4.1. *Let $G = (V, E)$ be an infinite line, i.e. $V = \mathbb{Z}$ and $xy \in E$ iff $|x - y| = 1$ for $x, y \in \mathbb{Z}$. We define the edge weight as $\mu_{xy} = 2^{1-\max\{|x|,|y|\}}$ for $xy \in E$. Obviously, $\mu(G) < \infty$. The function f defined as*

$$f(n) = \begin{cases} 2^n - 1, & n \geq 0, \\ 1 - 2^{-n}, & n < 0, \end{cases}$$

is a harmonic function on G. Noting that $f(n) = -f(-n)$, we have for any $q \in (0, 1)$

$$\|f\|_q^q = 2\sum_{n=1}^{\infty} f(n)^q (2^{-n+1} + 2^{-n}) \leq C \sum_{n=1}^{\infty} 2^{(q-1)n} < \infty.$$

Example 4.2. *Let $\Gamma_1 = (V_1, E_1)$ be any infinite graph with $\mu(\Gamma_1) = \infty$. Fix a vertex in Γ_1, say $p \in V_1$. Let $G = (V, E)$ be the weighted graph in the previous example and $\underline{0}$ the vertex representing the origin of $\mathbb{Z}$. We define a new graph $\Gamma = \Gamma_1 \wedge G$ by gluing the vertices p and $\underline{0}$ (i.e. identifying p with $\underline{0}$). Formally, $\Gamma = (V_\Gamma, E_\Gamma)$ where $V_\Gamma = (V_1 \setminus \{p\}) \cup V$ and $xy \in E_\Gamma$ iff $xy \in E_1$ for $x, y \in V_1 \setminus \{p\}$, or $xy \in E$ for $x, y \in V$, or $y = \underline{0}$ and $xp \in E_1$, or $x = y = \underline{0}$ and $pp \in E_1$. Since no new edges is added, we take the edge weights on Γ to be those of Γ_1 and G. Then*

$$g(x) = \begin{cases} 0, & x \in V_1 \setminus \{p\}, \\ f(x), & x \in V, \end{cases}$$

is a nonconstant L^q harmonic functions on Γ for $q \in (0, 1)$.

Remark 4.1. In both examples above, direct calculation shows that $\sum_{B_R} |f(x)|\mu_x = O(R)$. This means that (2) fails for $q = 1$.

Finally, as an application of Theorem 1.2, we study L^q $(1 \leq q < \infty)$ Liouville theorems for solutions to higher order operators on graphs. Let $\Delta^m := \Delta \circ \Delta \circ \cdots \circ \Delta$ be the m-fold composition of Laplace operators, i.e., the analogue of the polyharmonic operator in the continuous setting. A function $f \in \mathbb{R}^V$ is therefore called *polyharmonic* if $\Delta^m f = 0$ for some $m \geq 2$. The maximum principle is not available for polyharmonic functions. By our Theorem 1.2, we can prove the L^q $(1 \leq q < \infty)$ Liouville theorem for polyharmonic functions on graphs of infinite volume.

Theorem 4.1. *Let G be a graph of infinite volume, i.e. $\mu(G) = \infty$. If f is an L^q polyharmonic function on G for $1 \leq q < \infty$, then $f \equiv 0$.*

Proof. Suppose $\Delta^m f = 0$ for some $m \in \mathbb{N}$. We put $g^i := \Delta^i f$ for $1 \leq i \leq m$. Direct calculation (by Jensen's inequality) shows that the normalized Laplace operator is a bounded operator on L^q for any $q \in [1, \infty)$. Hence, we know that $g^i \in L^q$ by $f \in L^q$. Since $\Delta g^{m-1} = 0$ and $g^{m-1} \in L^q$, by Theorem 1.2, g^{m-1} is constant. Since $\mu(G) = \infty$, $g^{m-1} \equiv 0$. Iteratively, we get $f \equiv 0$. □

References

[CG98] T. Coulhon and A. Grigor'yan. Random walks on graphs with regular volume growth. *Geom. Funct. Anal.*, 8(4):656–701, 1998.

[Chu83] L. O. Chung. Existence of harmonic L^1 functions in complete Riemannian manifolds. *Proc. Amer. Math. Soc.*, 88(3):531–532, 1983.

[Chu97] Fan R. K. Chung. *Spectral Graph Theory (CBMS Regional Conference Series in Mathematics, No. 92)*. American Mathematical Society, 1997.

[Gri09] A. Grigor'yan. *Analysis on Graphs*. Lecture Notes, University Bielefeld, 2009.

[HS97] I. Holopainen and P. M. Soardi. A strong Liouville theorem for p-harmonic functions on graphs. *Ann. Acad. Sci. Fenn. Math.*, 22(1):205–226, 1997.

[Kar82] L. Karp. Subharmonic functions on real and complex manifolds. *Math. Z.*, 179(4):535–554, 1982.

[Li84] P. Li. Uniqueness of L^1 solutions for the Laplace equation and the heat equation on Riemannian manifolds. *J. Differential Geom.*, 20(2):447–457, 1984.

[Li12] P. Li. *Geometric analysis*, volume 134 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2012.

[LS84] P. Li and R. Schoen. Lp and mean value properties of subharmonic functions on Riemannian manifolds. *Acta Math.*, 153(3-4):279–301, 1984.

[LX10] Y. Lin and L. Xi. Lipschitz property of harmonic function on graphs. *J. Math. Anal. Appl.*, 366(2):673–678, 2010.

[Mas09] J. Masamune. A liouville property and its application to the laplacian of an infinite graph. In *Spectral analysis in geometry and number theory*, volume 484 of *Contemp. Math.*, pages 103–115, Providence, RI, 2009. Amer. Math. Soc.

[RSV97] M. Rigoli, M. Salvatori, and M. Vignati. Subharmonic functions on graphs. *Israel J. Math.*, 99:1–27, 1997.

[Stu94] K.-T. Sturm. Analysis on local Dirichlet spaces. I. Recurrence, conservativeness and L^p-Liouville properties. *J. Reine Angew. Math.*, 456:173–196, 1994.

[Yau76] S. T. Yau. Some function-theoretic properties of complete Riemannian manifold and their applications to geometry. *Indiana Univ. Math. J.*, 25(7):659–670, 1976.

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